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Solutions for MATHM1402 Exam

1. (a) State, without proof, the general formula for a Fourier series on $(-\pi, \pi)$ for a function $f(x)$, giving the expressions for the coefficients.

Solution: The Fourier series for $f(x)$ on $(-\pi, \pi)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

where the constants a_n and b_n are defined by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \end{aligned}$$

for $n = 0, 1, 2, \ldots$.

(b) Find the Fourier series for

$$f(x) = \begin{cases} 0 & , \quad \text{if } -\pi < x \leq -\frac{\pi}{2} \\ 1 & , \quad \text{if } -\frac{\pi}{2} < x \leq \frac{\pi}{2} \\ 0 & , \quad \text{if } \frac{\pi}{2} < x < \pi \end{cases}.$$

Solution: To find the Fourier series for f, we need to determine the constants a_n and b_n. We compute the b_n first.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin(nx)dx = 0$$

for all n. This could also be seen from the fact that f is an even function of x. Now we compute the a_n. For $n = 0$, we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dx = 1.$$

For $n \geq 1$, we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cos(nx)dx \\ &= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

If n is even, this is 0. So we are left only with the case when n is odd, i.e., $n = 2j-1$ for $j = 1, 2, \ldots$. Then

$$a_{2j-1} = \frac{2}{(2j-1)\pi}\sin\left(\frac{(2j-1)\pi}{2}\right) = \frac{2}{(2j-1)\pi}(-1)^{j-1}.$$

It follows that

$$f(x) = \frac{1}{2} + \sum_{j=1}^{\infty}\frac{2}{(2j-1)\pi}(-1)^{j-1}\cos\left((2j-1)x\right)$$

or

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\cos(x) - \frac{2}{3\pi}\cos(3x) + \frac{2}{5\pi}\cos(5x) - \frac{2}{7\pi}\cos(7x) + \ldots.$$

(c) Using Part b, or otherwise, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

Solution:] From Part b, we know that at $x = 0$, we have

$$f(0) = \frac{1}{2} + \frac{2}{\pi}\cos(0) - \frac{2}{3\pi}\cos(0) + \frac{2}{5\pi}\cos(0) - \frac{2}{7\pi}\cos(0) + \ldots,$$

which can be rewritten as

$$1 = \frac{1}{2} + \frac{2}{\pi} - \frac{2}{3\pi} + \frac{2}{5\pi} - \frac{2}{7\pi} + \ldots,$$

and again as

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

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(a) Define the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)},$$

where $x(u,v)$ and $y(u,v)$ are smooth functions.

Solution: The Jacobian is defined to be

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

(b) Using the definition from Part a, determine the Jacobian for the coordinate transformation defined by the functions

$$x(u,v) = \frac{u+v}{2} \quad \text{and} \quad y(u,v) = \frac{u-v}{2}.$$

Solution: In order to evaluate the Jacobian, we need to find the partial derivatives of x and y. We have

$$\frac{\partial x}{\partial u} = \frac{1}{2}, \frac{\partial y}{\partial v} = -\frac{1}{2}, \frac{\partial x}{\partial v} = \frac{1}{2}, \frac{\partial y}{\partial u} = \frac{1}{2}.$$

Therefore,

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}.$$

(c) Let R be the first quadrant in the xy-plane. Using the coordinate transformation from Part b, or otherwise, find

$$\iint_R e^{-(x+y)^2}\,dxdy.$$

Solution: We use the change of coordinates forumla:

$$\iint_R f(x,y)dxdy = \iint_{R^*} f(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

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We have $x + y = u$, so the integrand becomes e^{-u^2}. Since the region R is described by $x \geq 0$ and $y \geq 0$, the transformed region R^* satisfies $u + v \geq 0$ and $u - v \geq 0$, which is equivalent to the conditions $u \geq 0$ and $-u \leq v \leq u$. So we need to take this into account when we iterate the transformed integrals (i.e., v before u). We have

$$\begin{aligned}
\iint\limits_R e^{-(x+y)^2} dydx &= \iint\limits_{R^*} e^{-u^2} \left|-\frac{1}{2}\right| dudv \\
&= \frac{1}{2}\int_0^\infty \int_{-u}^{u} e^{-u^2} dvdu \\
&= \frac{1}{2}\int_0^\infty e^{-u^2} \left(\int_{-u}^{u} dv\right) du \\
&= \int_0^\infty e^{-u^2} udu = \frac{1}{2}
\end{aligned}$$

3. (a) State the Divergence Theorem carefully.

Solution: Given a surface S and a vector field $\boldsymbol{F}$ which is defined at all points on S and in its interior V, we have

$$\iint\limits_S \boldsymbol{F} \cdot \boldsymbol{n} dS = \iiint\limits_V \vec{\nabla} \cdot \boldsymbol{F} dV.$$

(b) Let S be the surface of the closed box defined by $0 \leq x \leq 4$, $0 \leq y \leq 2$, and $0 \leq z \leq 10$. Let a be a positive constant and consider the vector field

$$\boldsymbol{F}(x, y, z) = (3a^3 - 2a)x\boldsymbol{i} + \frac{x^2 \cos(z)}{z + 4}\boldsymbol{j} + \frac{\ln(|y + 1|)}{(x + 1)^2}\boldsymbol{k}.$$

Find the exact value of a so that the flux of $\boldsymbol{F}$ over S is 0.

Solution: Since $\boldsymbol{F}$ is defined at all points of S and V, we can apply the Divergence Theorem. As such, we need to find the divergence of $\boldsymbol{F}$. It is

$$\vec{\nabla} \cdot \boldsymbol{F} = 3a^3 - 2a + 0 + 0 = 3a^3 - 2a.$$

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It follows that

$$\iiint_V \vec{\nabla} \cdot \boldsymbol{F} dV = (3a^3 - 2a) \iiint_V dV = 80(3a^3 - 2a).$$

We want to find the value of $a > 0$ such that $80(3a^3 - 2a) = 0$. This occurs is

$$a(3a^2 - 2) = 0.$$

We can discard the value $a = 0$ since its non-positive. This leaves the condition $3a^2 - 2 = 0$, or $a = \pm\sqrt{2/3}$. We take $a = \sqrt{2/3}$, as this is the positive value of a for which the flux of $\boldsymbol{F}$ over S is 0.

(c) Considering the situation in Part b, how would the value of a change if we replace the vector field $\boldsymbol{F}$ by the vector field $\boldsymbol{F} + \boldsymbol{G}$, where $\boldsymbol{G}(x, y, z) = g_1(y, z)\boldsymbol{i} + g_2(x, z)\boldsymbol{j} + g_3(x, y)\boldsymbol{k}$ with smooth functions $g_1, g_2, g_3 : \mathbb{R}^2 \to \mathbb{R}$.

Solution: We note that

$$\vec{\nabla} \cdot (\boldsymbol{F} + \boldsymbol{G}) = \vec{\nabla} \cdot \boldsymbol{F} + \vec{\nabla} \cdot \boldsymbol{G}.$$

The form of $\boldsymbol{G}$ implies that $\vec{\nabla} \cdot \boldsymbol{G} = 0$. Since $\boldsymbol{G}$ is defined everywhere, the Divergence Theorem implies that the flux of $\boldsymbol{F} + \boldsymbol{G}$ over S is equal to the flux of $\boldsymbol{F}$ over S, and hence, the value of a remains unchanged, i.e., $a = \sqrt{2/3}$ results in the flux of $\boldsymbol{F} + \boldsymbol{G}$ over S being 0.

4. (a) State Green's Theorem in the plane carefully.

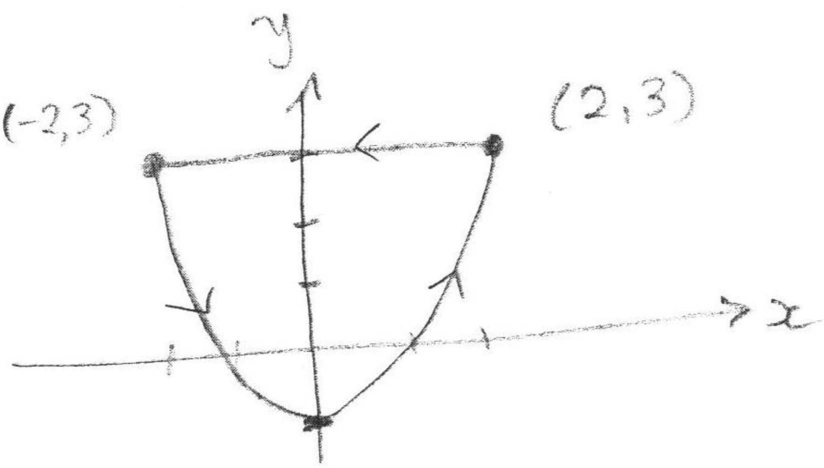
Solution: For a region R in the xy-plane with boundary C oriented in the anticlockwise direction and a smooth vector field $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j}$ defined on R and C, we have

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

(b) Sketch the contour C which is described as follows: Begin at the point $(2, 3)$. Go to the point $(-2, 3)$ along the straight line segment. Then go back to $(2, 3)$ along the curve given by the equation

$y = x^2 - 1$. This description also gives you the correct orientation of C.

Solution:



(c) Let $\boldsymbol{F}(x, y) = x\sin(x)\boldsymbol{i} + (xy + \ln(1 + y^2))\boldsymbol{j}$. Use Green's Theorem to calculate the circulation of $\boldsymbol{F}$ around C.

Solution: The circulation of $\boldsymbol{F}$ is given by $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$. Since $\boldsymbol{F}$ is defined everywhere in the plane, we can use Green's Theorem to pass to the double integral over the region R. We need the partial derivatives of F_1 and F_2. They are

$$\frac{\partial F_1}{\partial y} = 0 \quad \text{and} \quad \frac{\partial F_2}{\partial x} = x.$$

So

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_R x\,dx\,dy.$$

The region R is described by the inequalities $-2 \leq x \leq 2$ and

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$x^2 - 1 \le y \le 3$. Therefore.

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{-2}^{2}\int_{x^2-1}^{3} x\,dy\,dx$$
$$= \int_{-2}^{2} x(4 - x^2)dx$$
$$= \left[2x^2 - \frac{1}{4}x^4\right|_{-2}^{2} = 0.$$

The circulation of $\boldsymbol{F}$ around C is 0.

5. (a) State Stoke's Theorem carefully.

Solution: Given an oriented surface S with an oriented boundary C (such that the orientation of C is consistent with the right-hand rule) and a smooth vector field $\boldsymbol{F}$ defined on S and C, we have

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S (\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n} dS.$$

(b) Verify Stoke's Theorem for the vector field

$$\boldsymbol{F}(x, y, z) = -y\boldsymbol{i} + x\boldsymbol{j} + xz\boldsymbol{k}$$

and the surface S defined by $x^2 + y^2 + z^2 = 17$ and $z \ge 4$. Sketch the surface S.

Solution: A sketch of the surface S should show a cap on the sphere of radius 17 above the plane $z = 4$. To verify Stoke's Theorem, we need to compute both the circulation and flux integrals. We begin with the circulation integral.

The curve C is the intersection of the sphere of radius 17 centered at the origin with the plane $z = 4$, i.e., $x^2 + y^2 = 1$ with $z = 4$. We parameterize it as

$$\boldsymbol{r}(t) = \cos(t)\boldsymbol{i} + \sin(t)\boldsymbol{j} + 4\boldsymbol{k},$$

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for $0 \leq t \leq 2\pi$. Then

$$\begin{aligned} \boldsymbol{F} \cdot d\boldsymbol{r} &= \boldsymbol{F}\big(\boldsymbol{r}(t)\big) \cdot \boldsymbol{r}'(t)dt \\ &= (-\sin(t)\boldsymbol{i} + \cos(t)\boldsymbol{j} + 4\cos(t)\boldsymbol{k}) \cdot (-\sin(t)\boldsymbol{i} + \cos(t)\boldsymbol{j} + 0\boldsymbol{k} \\ &= \left(\sin^2(t) + \cos^2(t)\right) dt = dt. \end{aligned}$$

Therefore, the value of the circulation integral is

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^{2\pi} dt = 2\pi.$$

For the flux integral, we need the curl of $\boldsymbol{F}$. We have

$$\text{curl}\boldsymbol{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}.$$

Since $F_1 = -y$, $F_2 = x$, and $F_3 = xz$, this yields

$$\text{curl}\boldsymbol{F} = -z\boldsymbol{j} + 2\boldsymbol{k}.$$

So the flux integral becomes

$$\iint_S (\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n} dS = \iint_S (-z\boldsymbol{j} + 2\boldsymbol{k}) \cdot \boldsymbol{n} dS.$$

We use the function $f(x, y) = \sqrt{17 - x^2 - y^2}$ to give us the surface S as a graph over the plane, i.e., $z = f(x, y)$. The projection of S onto the xy-plane, which we call R, is the unit disc. Also,

$$\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{17 - x^2 - y^2}} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{17 - x^2 - y^2}}.$$

Then

$$\begin{aligned} \iint_S (-z\boldsymbol{j} + 2\boldsymbol{k}) \cdot \boldsymbol{n} dS &= \iint_R \left[-\frac{y}{\sqrt{17 - x^2 - y^2}}(-z\ ) + 2\right] dxdy \\ &= \iint_R [y + 2]\, dxdy \\ &= \iint_R y dxdy + 2\iint_R dxdy = 2\pi. \end{aligned}$$

This verifies Stoke's Theorem.

6. (a) Let $\boldsymbol{A}$ be a vector potential for $\boldsymbol{B}$, i.e., $\boldsymbol{B} = \text{curl}\boldsymbol{A}$. Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and show that

$$\boldsymbol{A} + \vec{\nabla}\phi$$

is also a vector potential for $\boldsymbol{B}$. Find an expression for the divergence of $\boldsymbol{A} + \vec{\nabla}\phi$ in terms of the divergence of $\boldsymbol{A}$.

Solution: We know that $\boldsymbol{B} = \text{curl}\boldsymbol{A}$. Since the curl of a gradient field is 0, we have

$$\text{curl}(\boldsymbol{A} + \vec{\nabla}\phi) = \text{curl}\boldsymbol{A} + \text{curl}\vec{\nabla}\phi = \text{curl}\boldsymbol{A} = \boldsymbol{B}.$$

Therefore, $\boldsymbol{A} + \vec{\nabla}\phi$ is also a vector potential for $\boldsymbol{B}$.
We take the divergence of $\boldsymbol{A} + \vec{\nabla}\phi$, i.e.,

$$\text{div}(\boldsymbol{A} + \vec{\nabla}\phi) = \text{div}\boldsymbol{A} + \text{div}\vec{\nabla}\phi = \text{div}\boldsymbol{A} + \Delta\phi.$$

So the divergence of $\boldsymbol{A} + \vec{\nabla}\phi$ is equal to the divergence of $\boldsymbol{A}$ plus the Laplacian of ϕ.

(b) For the vector potential $\boldsymbol{A} = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$, is it possible to find a smooth function $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\boldsymbol{A} + \vec{\nabla}\phi$ is divergence free? If so, provide a ϕ that works.

Solution: Using Part a, we seek a smooth function ϕ such that

$$\text{div}\boldsymbol{A} + \nabla^2\phi = 0.$$

We calculate that

$$\text{div}\boldsymbol{A} = 2 + 2 + 2 = 6.$$

As a result, we need a ϕ such that $\nabla^2\phi = -6$. There are infinitely many such functions to choose from, so we pick $\phi = -3x^2$. Then

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -6$$

as desired. Therefore, the vector potential $\boldsymbol{A} - 6x\boldsymbol{i}$ is divergence free.

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(c) A central vector field is one of the form $\boldsymbol{F} = f(r)\boldsymbol{r}$, where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, and $r = |\boldsymbol{r}|$. Show that any central vector field is irrotational, i.e., $\text{curl}\boldsymbol{F} = \boldsymbol{0}$.

Solution: We have $\boldsymbol{F} = f\left(\sqrt{x^2+y^2+z^2}\right)\left(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}\right)$. We compute each component of the curl separately.

$$\begin{aligned}\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} &= \frac{\partial}{\partial y}(zf(r)) - \frac{\partial}{\partial z}(yf(r)) \\ &= zf'(r)\frac{\partial r}{\partial y} - yf'(r)\frac{\partial r}{\partial z} \\ &= zf'(r)\frac{y}{r} - yf'(r)\frac{z}{r} = 0.\end{aligned}$$

$$\begin{aligned}\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} &= \frac{\partial}{\partial z}(xf(r)) - \frac{\partial}{\partial x}(zf(r)) \\ &= xf'(r)\frac{z}{r} - zf'(r)\frac{x}{r} = 0.\end{aligned}$$

$$\begin{aligned}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial x}(yf(r)) - \frac{\partial}{\partial y}(xf(r)) \\ &= yf'(r)\frac{x}{r} - xf'(r)\frac{y}{r} = 0.\end{aligned}$$

Therefore, $\text{curl}\boldsymbol{F} = \boldsymbol{0}$.